Proof of Bessel's Relation:$\frac{d}{dx}(x^{-v}J_v(x)) = -x^{-v}J_{v+1}(x)$*

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Claim: A Bessel function of higher order can be expressed by Bessel functions of lower orders

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\begin{aligned}
J_v(x): &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+v+n)}(\frac{x}{2})^{2n+v} \\
\frac{d}{dx}(x^{-v}J_v(x)) &= \frac{d}{dx}\{x^{-v}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+v+n)}(\frac{x}{2})^{2n+v}\} \\
&= \frac{d}{dx}\{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!\Gamma(1+v+n)2^{2n+v}}\} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n(2n)x^{2n-1}}{n!\Gamma(1+v+n)2^{2n+v}} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n(2n)x^{2n-1}}{n!(v+n)\Gamma(v+n)2^{2n+v}} \\
&= -x^{-v}\sum_{n=0}^{\infty}\frac{-(-1)^n(\frac{2n}{v+n})x^{2n+v-1}}{n!\Gamma(v+n)2^{2n+v}} \\
&= -x^{-v}\sum_{n=0}^{\infty}\frac{-(-1)^n(\frac{2n}{v+n})x^{2n+v-1}}{n!\Gamma(v+n)2^{2n+v}} \\
\text{(for n=0, the sum is 0)} &= -x^{-v}\sum_{n=1}^{\infty}\frac{-(-1)^n(\frac{2n}{v+n})x^{2n+v-1}}{n!\Gamma(v+n)2^{2n+v}} \\
&= -x^{-v}\sum_{n=1}^{\infty}\frac{-(-1)^n(\frac{2}{v+n})x^{2n+v-1}}{(n-1)!\Gamma(v+n)2^{2n+v}} \\
&= -x^{-v}\sum_{n=1}^{\infty}\frac{-(-1)^n x^{2n+v-1}}{(n-1)!(v+n)\Gamma(v+n)2^{2n+v-1}} \\
\text{(let n=k+1)} &= -x^{-v}\sum_{k=0}^{\infty}\frac{-(-1)^{k+1}x^{2(k+1)+v-1}}{(k)!(v+k+1)\Gamma(v+k+1)2^{2(k+1)+v-1}} \\
[-(-1)^{k+1}=(-1)^k] &= -x^{-v}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+v+1}}{(k)!(v+k+1)\Gamma(v+k+1)2^{2k+v+1}} \\
\text{(let k =n)} &= -x^{-v}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+v+1}}{(k)!(v+k+1)\Gamma(v+n+1)2^{2n+v+1}} \\
&= -x^{-v}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+v+1}}{k!\Gamma(v+n+2)2^{2n+v+1}} \\
&= -x^{-v}J_{v+1}(x)\ \square
\end{aligned}
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*made with LaTeX