Proof of Bessel's Relation: $\frac{2v}{x}J_v(x) - J_{v-1}(x) = J_{v+1}(x)$ *

Christopher Nofal

September 18, 2006

Claim: A Bessel function of higher order can be expressed by Bessel functions of lower orders.

$$
\begin{aligned}
J_v(x) &:= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+v+1)} (\frac{x}{2})^{2n+v} \\
J_{v-1}(x) &:= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+v)} (\frac{x}{2})^{2n+v-1} \\
\frac{2v}{x} J_v(x) - J_{v-1}(x) &= \sum_{n=0}^{\infty} \frac{2v}{x} \{ \frac{(-1)^n}{n!\Gamma(n+v+1)} (\frac{x}{2})^{2n+v} \} - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+v)} (\frac{x}{2})^{2n+v-1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n v}{n!\Gamma(n+v+1)} (\frac{x}{2})^{2n+v-1} - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+v)} (\frac{x}{2})^{2n+v-1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n v}{n!\Gamma(n+v+1)} (\frac{x}{2})^{2n+v-1} - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+v)} (\frac{x}{2})^{2n+v-1} \frac{(n+v)}{(n+v)} \\
\text{(note: for n=0, sum=0)} &= \sum_{n=0}^{\infty} (\frac{-(-1)^n n}{n!\Gamma(n+v+1)}) (\frac{x}{2})^{2n+v-1} \\
&= \sum_{n=1}^{\infty} (\frac{-(-1)^n}{(n-1)!\Gamma(n+v+1)}) (\frac{x}{2})^{2n+v-1} \\
\text{(let n=k+1)} &= \sum_{k=0}^{\infty} (\frac{-(-1)^{k+1}}{k!\Gamma(k+v+2)}) (\frac{x}{2})^{2k+v+1} \\
\text{(let k=n)} &= \sum_{k=0}^{\infty} (\frac{(-1)^n}{n!\Gamma(n+v+2)}) (\frac{x}{2})^{2n+v+1} \\
&= J_{v+1} \ \square
\end{aligned}
$$

*made with LaTeX